The JSE Zero-Coupon Yield Curves

Methodology Document

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Contents

1 Introduction ........................................................................................................................................ 3
2 The Daily JSE Zero Curves Report ................................................................................................... 4
3 The Bootstrapping Methodology ..................................................................................................... 4
  3.1 The Nominal Bond Curve ........................................................................................................... 4
  3.2 The Nominal Swap Curve .......................................................................................................... 5
  3.3 The Real Bond Curve ............................................................................................................... 6
4 Interpolation ..................................................................................................................................... 7
  4.1 Monotone Preserving \( r(t) \) Interpolation ................................................................................. 7
    4.1.1 The Interpolation Algorithm ............................................................................................. 8
    4.1.2 Extrapolation .................................................................................................................... 9
5 Input Instruments ........................................................................................................................... 9
  5.1 The Nominal Bond Curve .......................................................................................................... 9
    5.1.1 The Overnight Point ......................................................................................................... 9
    5.1.2 Treasury Bills .................................................................................................................. 10
    5.1.3 Government Bonds ....................................................................................................... 10
  5.2 The Nominal Swap Curve ........................................................................................................ 10
  5.3 The Real Bond Curve .............................................................................................................. 11
    5.3.1 The Overnight Point ....................................................................................................... 11
    5.3.2 Inflation Linked Government Bonds .............................................................................. 12

REFERENCES ...................................................................................................................................... 13
1 Introduction

The term structure of interest rates measures the relationship among the yields on zero-coupon bonds (often referred to as zero-coupon spot rates, or simply zero-coupon rates) of the same credit quality that differ only in their term to maturity. A yield curve is then a plot depicting the zero-coupon yield for a continuum of maturities, in some time interval. Yield curves have a number of roles to perform in the functioning of debt capital markets, including:

1. The valuation of any future cash flow (series of cash flows).
2. The calibration of various risk metrics particular to fixed income portfolios.
3. They give an important indication as to the market's expectation regarding future interest rates.
4. They are often analysed for the purpose of establishing fixed income trading strategies.
5. They are often used to calibrate no-arbitrage term structure models, like the models of Ho and Lee (1986), Hull and White (1990), and Cox et al. (1977).

In 2003 the Bond Exchange of South Africa, in conjunction with the Actuarial Society of South Africa launched the BEASSA Zero-Coupon Yield Curves. These curves represented a huge step forward in the provision of benchmarking and valuation tools for the South African Bond Market. The JSE, however, feels that the methodology underlying these curves are outdated, and that a new set of curves generated from a different methodology are in order. This document introduces the new JSE Zero-Coupon Yield Curves.

The JSE Zero-Coupon Yield Curves are a daily suite of three yield curves. One to cover the nominal bond market, one the nominal swaps market, and one to cover the inflation-linked bond market. Each curve will be a “perfect fit” curve, in the sense that each curve will exactly price back all of the base instruments. The aim of this document is to describe the methodology underlying the new set of curves, and to provide the criteria that will be used to select the input instruments to each curve. The structure of this document is as follows:

- Section 2 provides a brief summary of the structure of the daily zero curves excel workbook that will be distributed.
- Section 3 describes the bootstrapping methodology used when constructing each of the three curves.
- Section 4 provides an overview of the interpolation algorithm used when bootstrapping.
- Section 5 provides a description of the input instruments used in constructing each of the new zero-coupon yield curves. Furthermore, Section 5 provides the criteria used in selecting the input bonds used to calibrate the nominal and real bond curves.
2 The Daily JSE Zero Curves Report

As with the BEASSA curves, the daily JSE curves are available from the Exchange via email, and ftp download. The daily JSE Zero Curves email contains an Excel workbook with a name of the form “ZeroCurveyyyymmdd.xls”. It contains two worksheets:

1. Zeroes: This sheet shows nominal bond, nominal swap, and real bond zero-coupon yields (compounded as Nominal Annual Compounded Annually (NACA)) at daily points, from one day to fifteen thousand days.

2. Compact: This sheet is a compressed version of the “Zeroes” sheet. It shows the nominal bond, nominal swap, and real bond zero-coupon, and par yields (compounded in accordance with the convention for each specific market) at quarterly points, from one day to forty years.

3. Inputs: This sheet shows the inputs that were used to generate each of the three zero-coupon yield curves.

3 The Bootstrapping Methodology

When constructing a yield curve two decisions are crucial: we have to decide on a construction methodology, and we have to select a set of inputs to which our model can be calibrated. This section gives an overview of the bootstrapping methodology used to construct each of the JSE’s zero-coupon yield curves. This bootstrapping methodology is described in Hagan and West (2006), and Hagan and West (2008).

The bootstrapping methodology will converge to a discrete set of zero-coupon yields. If we interpolate these yields by using the same method of interpolation used for the bootstrap, we will obtain a yield curve function \( r(t) \), for \( t \in [0, \infty) \). Using this function, we can exactly price back all of the base instruments.

3.1 The Nominal Bond Curve

The rounded all-in price of a coupon paying bond, \( [\bar{A}] \), can be written as follows:

\[
[\bar{A}] = \sum_{i=1}^{n} p_i Z(t_0; t_{\text{settle}}, t_i),
\]

where:

- \( t_{\text{settle}} \) is the date on which the cash is actually delivered for the purchased bond.
- \( t_1, t_2, \ldots, t_n \) are the remaining cash flow dates associated with the particular bond.
- \( p_i \) is the cash flow occurring at \( t_i \), for \( i \in \{1, 2, \ldots, n\} \).
- \( Z(t_0; t_{\text{settle}}, t_i) \) is the discount factor observed at \( t_0 \), applicable from \( t_{\text{settle}} \) to \( t_i \), for \( i \in \{1, 2, \ldots, n\} \).
Equation (1) can be reorganised as follows:

\[
    r_n = \frac{1}{t_n} \left[ \ln p_n - \ln \left( \hat{A} \sum_{i=1}^{n-1} p_i e^{-r_i t_i} \right) \right],
\]

where \( r_i \) denotes the zero rate (continuously compounded) corresponding to \( t_i \), for \( i \in \{1,2, ..., n\} \). Hagan and West (2008) then postulate the following bootstrapping algorithm:

1. Choose a set of yield curve inputs, this should comprise of a set of money market inputs maturing at times \( \tau_1, \tau_2, ..., \tau_n \), and a set of coupon paying bonds maturing at times \( \tau_{\eta+1}, \tau_{\eta+2} \ldots \tau_m \).
2. Guess the values of \( r_{\eta+1}, r_{\eta+2} \ldots r_m \).
3. Apply an appropriate interpolation algorithm, and interpolate between \( \tau_1, \tau_2 \ldots \tau_m \) and \( r_1, r_2 \ldots r_m \) in order to estimate the zero rates corresponding to each cash flow date of each input instrument.
4. Insert the zero rates obtained in step 3 into equation (2) in order to obtain new estimates for \( r_{\eta+1}, r_{\eta+2} \ldots r_m \).
5. Repeat steps 3 and 4 until convergence is obtained.

### 3.2 The Nominal Swap Curve

Consider a just issued Forward Rate Agreement (FRA) spanning the period from \( t_1 \) to \( t_2 \). At inception we have that

\[
    R = \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1},
\]

where \( R \) represents the agreed fixed rate. It follows that

\[
    r_2 = \frac{R(t_2 - t_1) + r_1 t_1}{t_2}.
\]

Similarly, the fixed rate of a just issued \( n \)-year swap must satisfy:

\[
    R_n \sum_{i=1}^{n} \alpha_i Z_i = 1 - Z_n,
\]

where \( R_n \) denotes the agreed fixed rate, \( \alpha_i \) denotes the fraction of a year from \( t_{i-1} \) to \( t_i \), and \( Z_i \) denotes the discount factor from \( t_0 \) to \( t_i \), for \( i \in \{1,2, ..., n\} \). It follows that

\[
    r_n = \frac{-1}{t_n} \left[ \frac{1 - R_n \sum_{i=1}^{n-1} \alpha_i Z_i}{1 + R_n \alpha_n} \right].
\]
The following bootstrapping algorithm can then be applied to obtain the zero-coupon swap curve:

1. Choose a set of yield curve inputs, this should comprise of a set of money market inputs maturing at times $\tau_1, \tau_2, \ldots, \tau_q$, a set of FRA’s maturing at times $\tau_{q+1}, \tau_{q+2}, \ldots, \tau_m$, and a set interest rate swaps maturing at times $\tau_{m+1}, \tau_{m+2}, \ldots, \tau_\omega$.
2. Guess the values of $r_{q+1}, r_{q+2}, \ldots, r_\omega$.
3. Apply an appropriate interpolation algorithm, and interpolate between $\tau_1, \tau_2, \ldots, \tau_\omega$ and $r_1, r_2, \ldots, r_\omega$ in order to estimate the zero rates corresponding to each cash flow date of each input instrument.
4. Insert the zero rates obtained in step 3 into equations (4) and (6) in order to obtain new estimates for $r_{q+1}, r_{q+2}, \ldots, r_\omega$.
5. Repeat steps 3 and 4 until convergence is obtained.

3.3 The Real Bond Curve

The rounded all-in price of an inflation-linked bond can be written as follows:

$$[A]_{\text{inf}} = \Gamma \left( e^{\tilde{r}_{\text{settle}}t_{\text{settle}}} \sum_{i=1}^{n} p_i e^{-\tilde{r}_i t_i} \right)$$

where:

- $\Gamma$ represents the inflation factor for this particular bond on $t_{\text{settle}}$. $\Gamma$ is calculated by dividing the reference CPI corresponding to $t_{\text{settle}}$ (see Raffaelli (2006) for a description on how to calculate this quantity) by the reference CPI number for corresponding to the particular bond’s issue date.
- $p_i$ is the un-inflated cash flows occurring at $t_i$, for $i \in \{1, 2, \ldots, n\}$.
- $\tilde{r}_i$ is the real zero rate applicable from $t_0$ to $t_i$, for $i \in \{1, 2, \ldots, n\}$.

A similar algorithm to that used to construct the nominal bond curve is used to construct the real bond curve (see section 3.1).
4 Interpolation

As seen in the previous section, interpolation plays a crucial role in the construction of the JSE zero-coupon yield curves. Hagan and West (2006) notes that the task of deciding on an appropriate methodology for interpolating yield curve data is by no means trivial; some methods imply discontinuous forward rates whilst others imply negative forward rates. Discontinuous forward rates make no sense from an economic point of view (unless the discontinuities occur on MPC dates), whilst negative forward rates imply arbitrage opportunities. The JSE will make use of the interpolation algorithm described by du Preez (2011), § 6.

Consider the finite set of zero-coupon yields $r_1, r_2, ..., r_n$, for maturities $t_1, t_2, ..., t_n$. We would like to obtain a yield curve function $r(t)$, for $t \in [0, \infty)$, with properties:

- $r(t)$ should interpolate the data in the sense that $r(t_i) = r_i$, for $i \in \{1,2,...,n\}$.
- $r(t)$ should be continuous.
- The instantaneous forward rate curve; $f(t) := \frac{d}{dt} r(t)$, for $t \in [0, \infty)$ should be continuous.
- If $r_1, r_2, ..., r_n$ represents a set of nominal yields, then $r(t) t$, for $t \in [0, \infty)$ should be a monotone increasing function. This requirement will ensure that $f(t)$ is a positive function.

4.1 Monotone Preserving $r(t) t$ Interpolation

Consider applying a shape preserving cubic Hermite interpolation scheme, developed inter alia by Akima (1970), Fritsch and Carlson (1980), de Boor (1978, 2001), and Hyman (1983), to the function $r(t)t$, for $t \in [0, \infty)$. Our interpolation function takes the form:

$$r(t) t = a_i + b_i (t - t_i) + c_i (t - t_i)^2 + d_i (t - t_i)^3,$$

for $t_i \leq t \leq t_{i+1}$.

Assume that the values of the discrete set of instantaneous forward rates $f_1, f_2, ..., f_n$, for maturities $t_1, t_2, ..., t_n$ are known. Define $h_i := t_{i+1} - t_i$, and $m_i := \frac{r_i + r_{i+1} - 2f_i}{t_{i+1} - t_i}$, for $i \in \{0,1,2,...,n-1\}$, where the assumption underlying $m_0$ is that $r_0 = t_0 = 0$. The coefficients $a_i, b_i, c_i$ and $d_i$, for $i \in \{0,1,2,...,n-1\}$, can then be solved easily (see Hagan and West (2006)) to obtain:

- $a_i = r_i t_i$
- $b_i = f_i$
- $c_i = \frac{3m_i - b_i + 2b_i}{h_i}$
- $d_i = \frac{b_{i+1} + b_i - 2m_i}{h_i^2}$
The problem we face in practice is that the set of instantaneous forward rates \( f_1, f_2, \ldots, f_n \), will rarely be known, and we have to rely on a modelling technique. We will model \( f_i \), for \( i \in \{2, 3, \ldots, n-1\} \), as the slope at \( t_i \) of the quadratic that passes through \((t_{i+j}, m_{i+j})\), for \( j \in \{+1,0,-1\} \). We will have that:

\[
f_i = \frac{t_i - t_{i-1}}{t_{i+1} - t_{i-1}} m_i + \frac{t_{i+1} - t_i}{t_{i+1} - t_{i-1}} m_{i-1},
\]

for \( i \in \{2,3,\ldots,n-1\} \), whilst:

\[
f_1 = m_0, \\
f_n = m_{n-1}.
\]

These choices for \( f_1 \) and \( f_n \) will ensure that we do not have any discontinuities in the spot, and forward curves when performing “flat forward” extrapolation.

In its current form, our interpolation function has no mechanism which ensures that \( r(t)t \) is monotone increasing (if required). Hyman (1983) notes a simple generalization of what was recognized by de Boor (1978, 2001), namely that if \( r(t)t \) is locally increasing at \( t_i \), and if

\[
f_i \leq 3 \min(m_{i-1}, m_i),
\]

then \( r(t)t \) will be monotone increasing on the interval \([t_{i-1}, t_i]\), for \( i \in \{1,2,\ldots,n-1\} \). If required, we can thus force our interpolation function to produce a monotone increasing \( r(t)t \) function by clamping \( f_i \) as follows:

\[
f_i = \min\left(f_i, 3 \min(m_{i-1}, m_i)\right).
\]

for \( i \in \{2,3,\ldots,n-1\} \).

### 4.1.1 The Interpolation Algorithm

The following algorithm can then be applied to perform monotone preserving \( r(t)t \) interpolation:

- Calculate the value of \( i \) such that \( t_{i-1} \leq t \leq t_i \).
- Calculate the values of \( f_1, f_2, \ldots, f_n \), as described by equation (9).
- If negative forward rates are not allowed, clamp \( f_i \), for \( i \in \{2,3,\ldots,n-1\} \), as in equation (11).
- Calculate \( a_i, b_i, c_i \) and \( d_i \), for \( i \in \{0,1,2,\ldots,n-1\} \), from where \( r(t) \) can be calculated as

\[
r(t) = \frac{1}{t_i}(a_i + b_i(t - t_i) + c_i(t - t_i)^2 + d_i(t - t_i)^3).
\]
4.1.2 Extrapolation

Since $f_1' = f_2' = 0$, it would be reasonable to assume that $f(t) = f_1$, for $t \leq t_1$, and that $f(t) = f_n$, for $t \geq t_n$. It follows that

$$r(t) = \frac{1}{t} (f_1 t_1),$$

(13)

for $t \leq t_1$, whilst

$$r(t) = \frac{1}{t} (r_n t_n + f_n (t - t_n)),$$

(14)

for $t \geq t_n$.

5 Input Instruments

Selecting an appropriate set of inputs to which a yield curve should be calibrated can be a difficult exercise, and can be more of an art than a science. Hagan and West (2006) notes that by excluding too many instruments, one runs the risk of excluding valuable information, whilst by including too many instruments, one runs the risk of obtaining implausible shapes for the curve. This section gives a description of the inputs that the JSE intends to use for each of its zero-coupon yield curves.

5.1 The Nominal Bond Curve

The nominal zero-coupon bond curve represents the nominal zero-coupon yields at which the South African government can obtain funding. This curve is constructed through the use of Treasury Bills, and government bonds.

5.1.1 The Overnight Point

In South Africa, funds placed on call typically earn interest on an overnight basis, but capitalisation only occurs on the last day of each month. An investor leaving his/her money on call from time $t_0$ to time $\mathcal{D}$, where $\mathcal{D}$ is the last calendar day of the month in which $t_0$ falls, would thus receive $(1 + \bar{r} \alpha)$ at time $\mathcal{D}$, where $\alpha = \frac{\mathcal{D} - t_0}{365}$, and $\bar{r}$ is the average (arithmetic) overnight call rate observed from $t_0$ to $\mathcal{D}$. South African overnight call rates are thus not “true” overnight zero-coupon yields. Some modelling will thus be required; the JSE will employ a naive approach and assume that overnight call rates are “true” overnight zero-coupon yields. This assumption will be made for all the three JSE zero-coupon yield curves.
The SAFEX overnight rate will be the rate used to anchor the short end of the nominal bond curve. This rate represents the average rate that SAFEX receives on its deposits with the banks, weighted by the size of the investments placed at each bank. Strictly speaking, this rate does not have the same credit characteristics as the T-Bills and bonds that will be used to construct the rest of the curve; however, we will assume that the credit embedded in an overnight rate is negligible.

5.1.2 Treasury Bills
Given that 91, 182, 273 and 365-day Treasury Bills are auctioned weekly, offering transparent and visible pricing points, the JSE will use all of these Bills as input instruments.

5.1.3 Government Bonds
A yield curve measures the difference between zero coupon rates that differ only because of their term to maturity. The instruments used to calibrate a yield curve should thus have similar credit, and liquidity characteristics. The GOVI index is an index that measures the performance of the ten most liquid South African government bonds. The constituents of the GOVI index thus have similar credit, and liquidity characteristics, and as such, the inputs to the nominal bond curve on any given date, will be the constituents of the GOVI index on that specific date.

By limiting the constituents of the bond curve to the ten most liquid government bonds, the JSE runs the risk of ignoring meaningful market information. However, the JSE feels that the GOVI criterion represents a clean and effective framework for managing the constituents of the nominal bond curve.

5.2 The Nominal Swap Curve
The nominal swap curve represents the zero-coupon yields at which interbank funding can be placed or obtained. This curve will be constructed through the use of money market deposits, FRAs and swaps. The following table gives a description of the instruments used to construct the various sections of the nominal swap curve:

<table>
<thead>
<tr>
<th>Point</th>
<th>Instrument</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overnight</td>
<td>SAFEX Overnight</td>
</tr>
<tr>
<td>1 Month</td>
<td>1 month JIBAR</td>
</tr>
<tr>
<td>3 Month</td>
<td>3 month JIBAR</td>
</tr>
<tr>
<td>4 - 24 Months</td>
<td>FRAs</td>
</tr>
<tr>
<td>2 - 30 Years</td>
<td>Swaps</td>
</tr>
</tbody>
</table>

*Table 1: The inputs to the nominal swap curve*
Note, the “Inputs” worksheet of the daily zero-curve workbook will show the swaps and FRAs used to construct the nominal swap curve. The set of FRAs and swaps that will be used to construct the curve is expected to remain static for large periods of time; however, some minor adjustments are bound to occur from time to time. For example, the 2-year point of the curve is expected to be constructed through the use of the 2-year swap. Market circumstances may, however, at some future point in time, lead to the 21x24 FRA being instead of the 2-year swap.

5.3 The Real Bond Curve

The real bond curve represents the real zero-coupon yields which the South African government can obtain funding. This curve is constructed through the use of inflation linked money market instruments, and inflation linked government bonds.

5.3.1 The Overnight Point

The market for overnight inflation linked securities is non-existent. The overnight point on the real bond curve will thus have to be modelled. When modelling this point, careful consideration should be given to ensure that this point is consistent with the rest of the curve.

Real rates embedded in inflation linked bonds allow for lags in inflation accrual. In particular, cash flows are inflated by multiplying with the ratio of today’s 4-month lagged CPI number (see Raffaelli (2006) for a description of how to calculate this quantity), to the 4-month lagged CPI number observed on the bond’s issue date. If we apply the exact same methodology to an overnight deposit, then the redemption value of an overnight inflation linked zero-coupon bond would be:

\[ FV = \frac{CPI(t_0 + 1)}{CPI(t_0)}, \]

where \( CPI(t_0 + 1) \) denotes the 4-month lagged CPI number on \( t_0 + 1 \), and \( CPI(t_0) \) denotes the 4-month lagged CPI number on \( t_0 \). The present value of the zero-coupon bond is then

\[ PV = \frac{CPI(t_0 + 1)}{CPI(t_0)} \times \frac{1}{1 + r(1/365)} \]

where \( r \) is the nominal overnight rate (compounded simple). We would like to obtain the real overnight rate; \( \tilde{r} \), such that

\[ FV = \frac{1}{1 + \tilde{r}(1/365)} \]

It then follows that

\[ \tilde{r} = 365 \left[ \frac{CPI(t_0)(1 + r/365)}{CPI(t_0 + 1)} - 1 \right]. \]
A similar approach can be used to obtain the one month point on the real bond curve.

The problem with the approach described above is that the quantity $\frac{CPI(t_0)}{CPI(t_0+1)}$ is seasonal, and as result, will cause instability in the short-end of the real bond curve. The approach is, however, consistent with the pricing of inflation linked bonds. The JSE will thus use this methodology to model the overnight, and the one month point on the real bond curve. The modelled overnight real rate will be shown in the “Inputs” sheet of the daily zero curve report as the “1ddm (modelled)” rate. Similarly, the modelled one month rate will shown as the “1m (modelled)” rate.

5.3.2 Inflation Linked Government Bonds
A yield curve measures the difference between zero-coupon rates that differ only because of their term to maturity. The instruments used to calibrate a yield curve should thus have similar credit, and liquidity characteristics. The IGOV index is an index that measures the performance of the ten most liquid inflation-linked South African government bonds. The constituents of the IGOV index thus have similar credit, and liquidity characteristics, and as such, the inputs to the real bond curve on any given date, will be the constituents of the IGOV index on that particular date.
REFERENCES


